

UNCLASSIFIED

AD 402 914

*Reproduced
by the*

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION ALEXANDRIA, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63 3 3

DEPARTMENT OF THE NAVY
OFFICE OF THE CHIEF OF NAVAL OPERATIONS
WASHINGTON 25, D.C.

In Reply Refer To
(OEG)186-63
30 April 1963

402014

From: Director, Operations Evaluation Group
To: Distribution List

Subj: OEG Interim Research Memorandum 31; forwarding of

Encl: (1) OEG IRM-31 "A 2-Player N-Region Search Game"
Unclassified of 17 Jan 1963

1. Enclosure (1) is forwarded herewith for your information and retention.
2. This memorandum addresses itself to the problem: Given N regions with their associated conditional detection probabilities $\alpha_1, \dots, \alpha_N$, let player A choose one region to hide in, and let player B look in one region at a time until he finds A. The payoff, to player A, is the expected number of looks required of B to find A. The form of the optimal pure strategies for B is described, and the mixed extension of this game is shown to have a solution. Player B has a good strategy that is a mixture of at most N pure strategies. A numerical procedure for calculating the solution is given.
3. Attention is invited to the fact that this type of memorandum does not necessarily represent the opinion of the Operations Evaluation Group or of the U. S. Navy.
4. Additional copies of the IRM may be obtained from OEG.


JOSEPH H. ENGEL
Director

Operations Evaluation Group

DISTRIBUTION:
Attached List

DISTRIBUTION LIST FOR (OEG)186-63:

Dir, WSEG (2)

COM ASTIA (10)

DEPT OF NAVY ACTIVITIES

<u>SNDL</u>		<u>BUWEPS</u>	<u>BUSHIPS</u>	<u>ONR</u>
E4	DIRNRL	R	300	405
J60	USNA ANNA	R-12		493
J84	SUPTNAVPGSCOL (2)	R-14		
J95	PRESNAVWARCOL (2)	R-5		
W6B	BUWEPSREP APL/JHU			

MARINE CORPS ACTIVITIES

COMDTMARCORPS

Dir, MCEC

Dir, MCLFDC, Quantico

DEPT OF AIR FORCE ACTIVITIES

Dept of AF (8) (Attn: PDO 4008)

USAF Liaison Office, Rand Corp.

C/S, U.S. Air Force (Attn: AFX-PDB)

C/S, U.S. Air Force (Attn: AFCOA)

DEPT OF ARMY ACTIVITIES

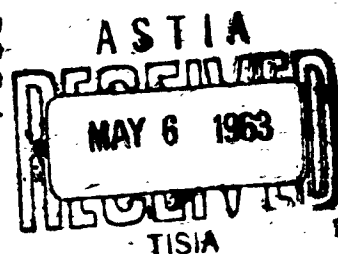
Dept of Army (Attn: C/R&D) (for RAC)

MISCELLANEOUS ACTIVITIES

SRI

OEG
~~~~~

This is an interim report of continuing research. It does not necessarily represent the opinion of OEG or the U. S. Navy. It may be modified or withdrawn at any time.



# **Interim Research Memorandum**

**OPERATIONS EVALUATION GROUP**

**Center for Naval Analyses**

**WASHINGTON 25, D. C.**

17 January 1963

INTERIM RESEARCH MEMORANDUM  
OPERATIONS EVALUATION GROUP

A 2-PLAYER N-REGION SEARCH GAME

By  
Joseph Bram  
IRM - 31

This is an interim report of continuing research. It does not necessarily represent the opinion of OEG or the U.S. Navy. It may be modified or withdrawn at any time.

INTERIM RESEARCH MEMORANDUM  
IRM-31

ABSTRACT

Given  $N$  regions with their associated conditional detection probabilities  $\alpha_1, \dots, \alpha_N$ , let player A choose one region to hide in, and let player B look in one region at a time until he finds A. The payoff, to player A, is the expected number of looks required of B to find A. The form of the optimal pure strategies for B is described, and the mixed extension of this game is shown to have a solution. Player B has a good strategy that is a mixture of at most  $N$  pure strategies. A numerical procedure for calculating the solution is given.

INTERIM RESEARCH MEMORANDUM  
IRM-31

A 2-PLAYER N-REGION SEARCH GAME

Suppose that player A can hide in any one of  $N$  regions, and that player B searches until he finds A. The probability that B can detect A in region  $i$ , given that A is there, is  $\alpha_i$ , and  $0 < \alpha_i < 1$ ,  $i = 1, \dots, N$ . The  $\alpha_i$  are known to both A and B. Player A chooses a region to hide in and stays there. Player B then looks in one region at a time until he finds A. The payoff is to player A; it is the expected number of looks that B must make until he finds A.

The class of pure strategies for B is the collection of all sequences  $y = \{y_1, y_2, \dots, y_v, \dots\}$  in which each  $y_v$  is either 1, or 2, or  $\dots$ , or  $N$ . Given such a sequence, B looks first in the region indicated by  $y_1$ , then in the region indicated by  $y_2$ , etc., until he finds A.

The set of A's pure strategies is the set  $\{1, 2, \dots, N\}$ ; he chooses an integer  $i$  and hides in region  $i$  until he is found by B.

For each  $i$  and each  $y$ , let  $M(i, y)$  denote the payoff to A. This is the expected number of looks required of B to find A. If  $y = \{y_1, y_2, \dots\}$ , we shall call the first  $k$  elements,  $y_1, \dots, y_k$  of  $y$  a segment of  $y$ ,  $k$  arbitrary. Its length is  $k$ . We define, for each  $y$ , and for each  $i, j$ ,  $1 \leq i \leq N$ ,  $1 \leq j < \infty$ ,

$c_{ij}(y)$  = the length of the smallest segment of  $y$  containing  $j$   $i$ 's,

if there is such a segment, and

$c_{ij}(y) = +\infty$

if no segment contains  $j$   $i$ 's.

For example, suppose  $N = 4$ , and

$y = \{2, 1, 3, 2, 4, 2, 4, 3, 3, 4, 1, 2, 1, 4, 3, 2, \dots\}$ .

Then, for  $j = 1, 2, 3, \dots$ , we have

$c_{1j} = 2, 11, 13, \dots$ ,

$c_{2j} = 1, 4, 6, 12, 16, \dots$ ,

$c_{3j} = 3, 8, 9, 15, \dots$ ,

$c_{4j} = 5, 7, 10, 14, \dots$ .



INTERIM RESEARCH MEMORANDUM  
IRM-31

We have, evidently, for each  $i$  and each  $y$ ,

$$M(i, y) = \sum_{j=1}^{\infty} c_{ij}(y) \alpha_i (1 - \alpha_i)^{j-1}. \quad (1)$$

(The sum of the series may be  $+\infty$ .)

If, in the long run, player A hides in region  $i$  with probability  $\xi_i$ ,  $\sum_1^N \xi_i = 1$ , we denote the expected value of  $M(i, y)$ , for fixed  $y$ , by  $M(\xi, y)$ .

We have

$$M(\xi, y) = \sum_{i=1}^N \xi_i M(i, y).$$

(Of course,  $0 \cdot \infty$  is defined by  $0 \cdot \infty = 0$ .)

We shall begin by showing that for each probability vector  $(\xi_1, \dots, \xi_N)$ , a mixed strategy for A, there is at least one optimal  $y(\xi)$ , i.e., a pure strategy for B such that

$$M(\xi, y(\xi)) \leq M(\xi, y)$$

for all  $y$ . Let us first observe that for each  $y$ , the set of finite values of  $c_{ij}(y)$  for  $i = 1, \dots, N$ ,  $j = 1, 2, \dots$  is the set of all positive integers, no two finite  $c_{ij}(y)$  are the same, and  $c_{i1} < c_{i2} < \dots$ . Conversely, if  $\{v_{ij}\}$  is a set of positive integers having the previous three properties, then there is a  $y$  such that

$$v_{ij} = c_{ij}(y),$$

$i = 1, \dots, N$ ;  $j = 1, 2, \dots$ . In our example above, the knowledge of all the  $c_{ij}$  enables us to reconstruct the strategy  $y$ ; from  $c_{1j}$  we see that we must look in region 1 on the 2<sup>nd</sup>, 11<sup>th</sup>, 13<sup>th</sup>, ... looks, in region 2 on the 1<sup>st</sup>, 4<sup>th</sup>, 6<sup>th</sup>, ..., looks, etc. This remark will be useful later.

INTERIM RESEARCH MEMORANDUM  
IRM-31

For each  $y$ , we now have

$$M(\xi, y) = \sum_{i=1}^N \sum_{j=1}^{\infty} c_{ij}(y) \xi_i \alpha_i (1 - \alpha_i)^{j-1}. \quad (2)$$

Since the  $c_{ij}$ 's run through all the positive integers, we can write

$$M(\xi, y) = \sum_{v=1}^{\infty} v \omega_v(y) + \sum_{i,j} \infty \cdot \xi_{ij}(y) \quad (3)$$

in which the  $\omega_v$  are weights from the set of all numbers  $\xi_i \alpha_i (1 - \alpha_i)^{j-1}$ , for which  $c_{ij}$  is finite,  $i = 1, \dots, N$ ;  $j = 1, 2, \dots$ , and  $\xi_{ij}$  runs through the  $\xi_i \alpha_i (1 - \alpha_i)^{j-1}$  for which  $c_{ij} = \infty$ . (In fact, if  $v = c_{ij}(y)$ , then

$\omega_v(y) = \xi_i \alpha_i (1 - \alpha_i)^{j-1}$ .) We have  $\sum \omega_v(y) + \sum \xi_{ij} = 1$ . Furthermore, if  $y'$  is another strategy, then  $\omega_v(y')$  and  $\xi_{ij}(y')$  are obtained simply by

rearranging the weights  $\{\xi_i \alpha_i (1 - \alpha_i)^{j-1}\}$ . The optimal  $y(\xi)$  can now be described very easily. We need only recall that if  $\{F_y(v)\}$  is a collection of (cumulative) distribution functions on the positive  $v$ -axis, with  $F_y(0) = 0$ ,

$F_y(\infty) = 1$ , and if for some  $y$ ,  $\int_0^{\infty} v dF_y(v) < \infty$ , then

$$\begin{aligned} E(v) &= \int_0^{\infty} v dF_y(v) \\ &= - \int_0^{\infty} v d[1 - F_y(v)] \\ &= \int_0^{\infty} [1 - F_y(v)] dv \end{aligned}$$

INTERIM RESEARCH MEMORANDUM  
IRM-31

(since  $v \left[ 1 - F_y(v) \right] \leq - \int_v^\infty v d(1 - F_y(v)) \rightarrow 0$  as  $v \rightarrow \infty$ ).

Therefore, if for some  $y_0$ , we have

$$F_{y_0}(v) \geq F_y(v)$$

for all  $v$  and for all  $y$ , then  $\int_0^\infty v dF_{y_0}(v)$  is the minimum with respect

to  $y$ . In our situation, we take, for each  $y$ ,  $F_y(v_0) = \sum_{v=1}^{v_0} \omega_v(y)$  for each  $v_0$ .

(Note: There is a  $y$  for which  $E(v) < \infty$ , e.g.,  $y = (1, 2, \dots, N, 1, 2, \dots, N, \dots, 1, 2, \dots, N, \dots)$ .) Then an optimal  $y = y(\xi)$  is obtained by taking

$\omega_1$  to be the largest of  $\{\xi_i \alpha_i (1 - \alpha_i)^{j-1}\}$ ,  $\omega_2$  the next largest, etc. In case there are two or more weights remaining that are equal, we can choose any of them for the next  $\omega_v$ . We shall refer to such a situation as a "tie". When the  $\omega_v$ 's have been determined, our optimal policy is also. We summarize the result in the following:

Lemma 1: If  $(\xi_1, \dots, \xi_N)$  is a mixed strategy for A, then an optimal pure strategy  $y = y(\xi)$  for B can be obtained as follows: Take all the numbers  $\xi_i \alpha_i (1 - \alpha_i)^{j-1}$ ,  $i = 1, \dots, N$ ;  $j = 1, 2, \dots$  and order them in a decreasing sequence,  $\{\omega_v\}$ . If  $\omega_v = \xi_i \alpha_i (1 - \alpha_i)^{j-1}$ , then the  $v^{\text{th}}$  look must be in region  $i$ .

In the following, we shall restrict player B to those pure strategies  $y$  for which

$$M(i, y) < \infty,$$

$i = 1, \dots, N$ . We shall see that in the mixed extension of our game, this is no real restriction on B.

Let us now proceed to the mixed extension of our game. The mixed strategies for A are the probability vectors  $(\xi_1, \dots, \xi_N)$ ,

$\xi_i \geq 0$ ,  $\sum_{i=1}^N \xi_i = 1$ . The class of mixed strategies for B is the collection

INTERIM RESEARCH MEMORANDUM  
IRM-31

of all functions  $\eta = \eta(y)$  defined on the set of pure strategies  $y$  for B, such

that  $\eta(y) = 0$  except for finitely many  $y$ 's,  $\sum_y \eta(y) = 1$  (i.e., B is allowed to

mix over any finite number of  $y$ 's). Such a game, in which player A has a finite number of pure strategies, is called an S-game, cf. [1]. The payoff is denoted by  $M(\xi, \eta)$  :

$$M(\xi, \eta) = \sum_{i=1}^N \xi_i \sum_y \eta(y) M(i, y).$$

The relevant theorem in this situation is as follows [1; 49] :

Theorem 1: Every S-game has a value  $v$ . Player A has a good strategy  $\xi^*$ , i.e.,

$$v \leq M(\xi^*, y)$$

for every  $y$ . For every  $\epsilon > 0$ , there is an  $\eta$  such that

$$M(i, \eta) < v + \epsilon,$$

$i = 1, 2, \dots, N$  (where  $M(i, \eta) = \sum_y \eta(y) M(i, y)$ ).

That an S-game always has a value  $v$  means that

$$\inf_{\eta} \sup_{\xi} M(\xi, \eta) = \sup_{\xi} \inf_{\eta} M(\xi, \eta) = v.$$

The main task remaining before us is to prove that player B has a good strategy. The theorem tells us only that B has an " $\epsilon$ -good" strategy for every  $\epsilon > 0$ .

The discussion of S-games is based on the following construction. For each  $y$ , let  $s(y)$  be the point in  $R_N$  (real  $N$ -space) with coordinates  $M(i, y)$ ,  $i = 1, \dots, N$ :

INTERIM RESEARCH MEMORANDUM  
IRM-31

$$s(y) = \begin{bmatrix} M(1, y) \\ \vdots \\ M(N, Y) \end{bmatrix} .$$

Let  $S$  denote the set of all  $s(y)$ 's in  $R_N$ . Let  $S^*$  denote the convex hull of  $S$ , i.e., the set of all finite convex combinations of points in  $S$ .

For any two vectors  $\xi$  and  $u$  in  $R_N$ , we write the inner product of  $\xi$  and  $u$  thus:

$$(\xi, u) = \sum_{i=1}^N \xi_i u_i .$$

Now, if  $\eta$  is a mixed strategy for B, then  $\sum_y \eta(y) s(y)$  belongs to  $S^*$ .

Conversely, if  $u$  is in  $S^*$ , then there is an  $\eta = \eta(y)$  such that  $u = \sum_y \eta(y) s(y)$ .

It follows that our  $S$ -game is equivalent to the following: player A chooses a probability vector  $\xi$ , player B chooses a point  $u$  in  $S^*$ , and the payoff is

$$M = (\xi, u) .$$

Let us note that in our case the set  $S$ , therefore also  $S^*$ , lies in the positive orthant of  $R_N$  -- all points have positive coordinates.

We shall briefly sketch one way to prove Theorem 1. Define, for each real  $\gamma$ , the open orthant

$$T_\gamma = \{u : u_i < \gamma, i = 1, 2, \dots, N\} .$$

Let

$$\gamma^* = \sup \{ \gamma : T_\gamma \cap S^* \neq \emptyset \} .$$

INTERIM RESEARCH MEMORANDUM  
IRM-31

Then, if we write  $Cl(E)$  for the closure of any set  $E$  in  $R_N$ , we have

$$Cl(T_{\gamma*}) \cap Cl(S^*) \neq \emptyset.$$

Let  $u^* \in Cl(T_{\gamma*}) \cap Cl(S^*)$ . Let

$$(\xi^*, u) = v$$

be the equation of a hyperplane separating  $T_{\gamma*}$  and  $S^*$ , with  $\sum_{i=1}^N \xi_i^* = 1$ .

Then

$$a) \quad \xi_i^* \geq 0, \quad i = 1, \dots, N,$$

$$b) \quad (\xi^*, u^*) = v,$$

$$c) \quad (\xi^*, u) \geq v \text{ for all } u \text{ in } S^*,$$

$$d) \quad (\xi^*, u) \leq v \text{ for all } u \text{ in } T_{\gamma*}$$

$$e) \quad \gamma^* = v,$$

$$f) \quad \max_{1 \leq i \leq N} u_i^* = v.$$

$$g) \quad \text{For each } i, \quad 1 \leq i \leq N \quad \text{either } \xi_i^* = 0 \text{ or else } u_i^* = v.$$

Once these facts have been established, we set

$$v_L = \sup_{\xi} \inf_{u \in S^*} (\xi, u),$$

$$v_U = \inf_{u \in S^*} \sup_{\xi} (\xi, u),$$

in which  $\xi$  ranges through the set of probability vectors. Then  $v_L \leq v_U$ ,

INTERIM RESEARCH MEMORANDUM  
IRM-31

of course. Now

$$v_L \geq \inf_{u \in S^*} (\xi, u) \text{ for all } \xi.$$

Therefore

$$v_L \geq \inf_{u \in S^*} (\xi^*, u).$$

Since  $u^*$  is a limit point of  $S^*$ , we have from c) and b) above,

$$v_L \geq v. \quad (3)$$

Also,

$$v_U \leq \sup_{\xi} (\xi, u) \text{ for all } u \text{ in } S^* ;$$

therefore

$$v_U \leq \max_{1 \leq i \leq N} u_i \text{ for all } u \text{ in } S^*,$$

and since  $u^* \in Cl(S^*)$ , we have also

$$v_U \leq \max_{1 \leq i \leq N} u_i^* = v \quad (4)$$

by f). Now (3) and (4) show that  $v_L = v_U = v$ .

Then  $\xi^*$  is a good strategy for  $A$ , using c). Also, if  $\epsilon > 0$ , there is a  $u$  in  $S^*$  such that

$$|u_i - u_i^*| < \epsilon$$

for  $i = 1, \dots, N$ , so that

$$u_i < v + \epsilon;$$

$u$  is of the form

$$u = \sum_y \eta(y) s(y)$$

for some  $\eta$ , and  $\eta$  is an " $\epsilon$ -good" strategy. This completes the proof of Theorem 1.

We shall prove that  $u^*$  belongs to  $S^*$ ; then  $u^*$  will be a convex combination

$$u^* = \sum_y \eta^*(y) s(y)$$

of points in  $S$ , and  $\eta^*$  will be a good strategy for  $B$ . First we require two more lemmas.

Lemma 2: Let  $u^0$  be a boundary point of  $S^*$ , and suppose there is a sequence  $\{s(y_n)\}$  in  $S$  converging to  $u^0$ . Let  $(\xi, u) = \gamma$ , with

$\sum_{i=1}^N \xi_i = 1$ , be the equation of a supporting hyperplane through  $u^0$ . Then

$\xi_i > 0$  for  $i = 1, \dots, N$ , and there is a  $y^0$  such that  $u^0 = s(y^0)$ , i.e.,  $u^0 \in S$ .

Proof: It is clear, in the first place, that  $\xi_i \geq 0$ ,  $i = 1, \dots, N$ , since if, for example,  $\xi_1 < 0$ , then by taking a sequence of pure strategies  $y^n$  in which player  $B$  looks less and less frequently in region 1, we will have

$$(\xi, s(y^n)) \rightarrow -\infty,$$

contradicting

$$(\xi, u) \geq \gamma \geq 0 \text{ for all } u \text{ in } S^*.$$



INTERIM RESEARCH MEMORANDUM  
IRM-31

Let  $u^0$  have coordinates  $u_i^0$ . Since  $s(y_n) \rightarrow u^0$ , we have, if  $\epsilon > 0$ ,

$$M(i, y_n) = \sum_{j=1}^{\infty} c_{ij}(y_n) \alpha_i (1 - \alpha_i)^{j-1} < u_i^0 + \epsilon,$$

$i = 1, \dots, N$ , for all  $n$  large enough. Therefore, for each  $i$  and  $j$ ,

$$0 < c_{ij}(y_n) < (u_i^0 + \epsilon) / \alpha_i (1 - \alpha_i)^{j-1}$$

for all  $n$  large enough, i.e., for fixed  $i, j$ , the sequence  $\{c_{ij}(y_n)\}$  is bounded.

Therefore, for a subsequence of  $\{y_n\}$ ,  $c_{11}(y_n)$  converges; for a subsequence of the latter,  $c_{21}(y_n)$  converges, etc., and by the "diagonal" procedure, we obtain a subsequence, call it  $\{y_n\}$  again, for which  $c_{ij}(y_n)$  converges for every  $ij$ . Let

$$\lim_{n \rightarrow \infty} c_{ij}(y_n) = \gamma_{ij},$$

$i = 1, \dots, N; j = 1, 2, \dots$ . It is clear that for each  $i, j$ ,  $\gamma_{ij}$  is a positive integer, all the  $\gamma_{ij}$  are distinct, and  $\gamma_{ij} < \gamma_{ik}$  if  $j < k$ . Also, every integer  $v$  appears among the  $\gamma_{ij}$ . Indeed, for each  $n$ , the number  $v$  must appear somewhere among the  $vN$  numbers  $c_{ij}(y_n)$ ,  $i = 1, \dots, N; j = 1, \dots, v$ . Since we have  $c_{ij}(y_n) = \gamma_{ij}$  for all  $n$  large enough for  $i = 1, \dots, N$ ,  $j = 1, \dots, v$ , it follows that  $v$  is equal to some  $\gamma_{ij}$ .

From our previous remarks (before Lemma 1), we see that  $\{\gamma_{ij}\}$  comes from some pure strategy  $y^0$ ,

$$\gamma_{ij} = c_{ij}(y^0).$$

For each integer  $q$ , we have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^q c_{ij}(y_n) \alpha_i (1 - \alpha_i)^{j-1} = \sum_{j=1}^q \gamma_{ij} \alpha_i (1 - \alpha_i)^{j-1} \leq u_i^0.$$

Therefore,

$$\sum_{j=1}^{\infty} \gamma_{ij} \alpha_i (1 - \alpha_i)^{j-1} \leq u_i^0, \quad (5)$$

$i = 1, \dots, N$ , or

$$M(i, y^0) \leq u_i^0. \quad (6)$$

It follows, since  $\xi_i \geq 0$ , that

$$(\xi, s(y^0)) \leq (\xi, u^0) = \gamma,$$

and therefore

$$(\xi, s(y^0)) = \gamma \quad (7)$$

(because  $(\xi, u) \geq \gamma$  for  $u$  in  $S^*$ ). This means that  $y^0$  is optimal against  $\xi$ , and from Lemma 1, we see that every  $\xi_i > 0$ ; indeed, if  $\xi_1 = 0$ , say, then the optimal strategy  $y^0$  would require no looks in region 1, contradicting  $M(1, y^0) < \infty$ . But if  $\xi_i > 0$  for  $i = 1, \dots, N$ , then we must have had equality in (6) for  $i = 1, \dots, N$  in order for (7) to be valid. This means that  $s(y^0) = u^0$ , and the proof is complete.

**Lemma 3:** Let  $E$  be a closed convex subset of  $R_N$  lying in the positive orthant (all coordinates  $u_i$  of  $u$  in  $E$  are non-negative). Then  $E$  contains an extreme point.

**INTERIM RESEARCH MEMORANDUM**  
**IRM-31**

Proof: Let

$$\psi(u) = u_1 + \dots + u_N,$$

and let

$$c = \inf_{u \in E} \psi(u).$$

The set

$$G = \{u \in E: c \leq \psi(u) \leq c + 1\}$$

is closed, bounded, and non-empty. Therefore there is a  $u^0$  in  $G$  for which  $\psi(u^0) = c$ .

Let

$$H = \{u \in E: \psi(u) = c\}.$$

Then  $H$  is non-empty, closed, bounded, and convex. Therefore  $H$  contains an extreme point  $u^*$ . This point  $u^*$  is also an extreme point of  $E$ . (If  $u^* = (u^1 + u^2)/2$  with  $u^1, u^2$  in  $E$ , then  $\psi(u^*) = (\psi(u^1) + \psi(u^2))/2 = c$ ; since  $\psi(u^1) \geq c$  and  $\psi(u^2) \geq c$ , we have  $\psi(u^1) = \psi(u^2) = c$ ,  $u^1, u^2$  belong to  $H$ .)

Theorem 2:  $S^*$  is closed.

Proof: Let  $u^0$  be a boundary point of  $S^*$ , and let

$$(\xi, u) = \gamma$$

be the equation of a supporting hyperplane through  $u^0$  with  $\sum_{i=1}^N \xi_i = 1$ .

Then

$$(\xi, u) \geq \gamma$$

for all  $u$  in  $S^*$ , and  $\xi_i \geq 0$ ,  $i = 1, \dots, N$ , as indicated in the proof of Lemma 1.

Let  $J$  be the hyperplane just described, and set

$$E = J \cap C1(S^*) .$$

Since  $E$  is closed and convex, and lies in the positive orthant of  $R_N$ ,  $E$  contains an extreme point  $u^*$ , by Lemma 3. Then  $u^*$  must also be an extreme point of  $C1(S^*)$ .

Now every  $u$  in  $S^*$  is a convex combination of at most  $N + 1$  points of  $S$  [1, 36]:

$$u = \sum_{j=0}^N \lambda_j s^j .$$

At least one of the  $\lambda_j$  must be  $\geq \frac{1}{N+1}$ . If we lump the remaining  $s^j$  together, we can write, for every  $u$  in  $S^*$ :

$$u = \alpha s + (1 - \alpha) v$$

with  $s$  in  $S$ ,  $v$  in  $S^*$ ,  $1/(N+1) \leq \alpha \leq 1$ . Now let  $u^n \rightarrow u^*$ ,

$$u^n = \alpha_n s^n + (1 - \alpha_n) v^n, \tag{8}$$

$s^n$  in  $S$ ,  $v^n$  in  $S^*$ ,  $1/(N+1) \leq \alpha_n \leq 1$ .

It follows that on the line joining  $s^n$  and  $v^n$ , there is a point  $w^n$  such that

$$u^n = s^n/(N+1) + N w^n/(N+1), \tag{9}$$

$s^n$  in  $S$ ,  $w^n$  in  $S^*$ . Let

$$e = \overbrace{(1, 1, \dots, 1)}^N .$$

INTERIM RESEARCH MEMORANDUM  
IRM-31

Since  $u^n \rightarrow u^*$ , we have

$$(e, u^n) = (e, s^n) / (N+1) + N(e, w^n) / (N+1) \rightarrow (e, u^*).$$

Since  $s^n$  and  $w^n$  have positive coordinates, it follows that  $(e, s^n)$  is bounded; so is  $\{s^n\}$ . Therefore, a subsequence converges and we may suppose  $s^n \rightarrow s^*$ ; by (9),  $w^n \rightarrow w^*$ ,

$$u^* = s^* / (N+1) + N w^* / (N+1).$$

Since  $u^*$  is an extreme point of  $Cl(S^*)$ , we have

$$u^* = s^* = w^*,$$

i.e.,  $u^* = \lim_n s^n$ ,  $s^n$  in  $S$ . By Lemma 2, we conclude that  $u^* = s(y^0)$  for some  $y^0$ , and that the hyperplane through  $u^*$ , passing also through our original  $u^0$ , whose equation was

$$J: (\xi, u) = \gamma,$$

must be such that  $\xi_i > 0$  for  $i = 1, \dots, N$ . From this, we see that our set  $E = J \cap Cl(S^*)$  is bounded as well as closed.

We have shown that every extreme point of  $E$  belongs to  $S$ . Since  $E$  is compact,  $E$  is generated by its extreme points. Therefore our original boundary point  $u^0$ , which belongs to  $E$ , is a convex combination of extreme points of  $E$ , i.e., of points of  $S$ , so that  $u^0 \in S^*$ . The proof is now complete.

Corollary: The mixed extension of the original game between  $A$  and  $B$  in which  $B$  can mix over all of his pure strategies (even if  $M(i, y) = \infty$  for some  $i$ ) has a solution; it coincides with the solution of the modified game ( $M(i, y) < \infty$ ,  $i = 1, \dots, N$ ). Player  $B$  has a good strategy  $\eta^*$  that is a mixture over at most  $N$  pure strategies. We have  $\sum_y \eta^*(y) M(i, y) = v$ ,  $i = 1, 2, \dots, N$ .

# INTERIM RESEARCH MEMORANDUM

IRM-31

**Proof:** It is clear from the preceding that if  $\xi^*$  is a good strategy for A (cf. Theorem 1), then  $\xi_i^* > 0$ ,  $i = 1, \dots, N$ . We have seen that  $S^*$  is closed, so that the point  $u^* \in Cl(S^*) \cap T_v$ , described in the discussion of Theorem 1, belongs to  $S^*$ ;

$$u^* = \sum_y \eta^*(y) s(y)$$

for some  $\eta^*$ . We have

$$(\xi, u^*) \leq v \leq (\xi^*, u) \quad (10)$$

for every probability vector  $\xi$  and every  $u$  in  $S^*$ . Since  $\xi_i^* > 0$  for  $i = 1, \dots, N$ , we have  $u_i^* = v$  for  $i = 1, \dots, N$ , from g) (preceding (3)), i.e.,

$$\sum_y \eta^*(y) M(i, y) = v,$$

$i = 1, \dots, N$ . Equation (10) implies that

$$v \leq (\xi^*, s(y)) \quad (11)$$

for every pure strategy  $y$  for which  $s(y)$  is finite. Since  $\xi_i^* > 0$ ,  $i = 1, \dots, N$ , (11) is a fortiori true if  $y$  is a pure strategy for which  $M(i, y) = \infty$  for some  $i$ .

Since

$$(\xi^*, s(y)) = v$$

for every  $y$  for which  $\eta^*(y) > 0$  (otherwise we'd have  $(\xi^*, u^*) > v$ ),  $u^*$  is a convex combination of points  $s(y)$  of  $S$  lying in an  $N - 1$  dimensional space.

Hence  $[1, 36]$ ,  $u^*$  can be expressed as a convex combination of at most  $N$  of these  $s(y)$ 's, and the proof is complete.

# INTERIM RESEARCH MEMORANDUM

IRM-31

Let  $\Sigma$  denote the closed simplex of all probability vectors  $\xi$ ,

$$\xi_i \geq 0, \sum_{i=1}^N \xi_i = 1.$$

Let

$$F(\xi) = \inf_y (\xi, s(y)). \quad (12)$$

By Lemma 1, there is a  $y$  that is optimal against  $\xi$ , so that we can write

$$F(\xi) = \min_y (\xi, s(y)) = (\xi, s(y(\xi))),$$

in which  $y(\xi)$  is some pure strategy optimal against  $\xi$ .

Theorem 3: 1°  $F(\xi)$  is concave on  $\Sigma$ ,

2°  $F(\xi)$  is continuous on  $\Sigma$ ,

3°  $\max_{\xi} F(\xi) = v$ ,

4°  $F(\xi)$  achieves its maximum

at an interior point of  $\Sigma$ .

Proof: 1° By (12),  $F(\xi)$  is the infimum of a collection of linear functions of  $\xi$ , all bounded below, by zero.

$$3^\circ \max_{\xi} F(\xi) = \max_{\xi} \inf_{u \in S^*} (\xi, u) = v.$$

$$4^\circ \text{ If } F(\xi^*) = \max_{\xi} F(\xi), \text{ then}$$

$v = F(\xi^*) = (\xi^*, y(\xi^*))$ , and from the preceding discussion (e.g., Lemma 2)  $\xi_i^* > 0$ ,  $i = 1, \dots, N$ .

# INTERIM RESEARCH MEMORANDUM

IRM-31

2° Since  $F(\xi) = \inf_y (\xi, s(y))$ , bounded below, and each  $(\xi, s(y))$  is continuous in  $\xi$ ,  $F(\xi)$  is upper semi-continuous. Therefore, whenever  $\xi^n \rightarrow \xi^0$ , we have

$$\overline{\lim}_{n \rightarrow \infty} F(\xi^n) \leq F(\xi^0). \quad (13)$$

Also,  $F(\xi)$  is concave, so that if  $\xi^n \rightarrow \xi^0$ , then also  $(\xi^n + \xi^0)/2 \rightarrow \xi^0$  and

$$F((\xi^n + \xi^0)/2) \geq (F(\xi^n) + F(\xi^0))/2.$$

By (13),

$$F(\xi^0) \geq \overline{\lim}_{n \rightarrow \infty} F(1/2(\xi^n + \xi^0)) \geq 1/2 \overline{\lim}_{n \rightarrow \infty} F(\xi^n) + 1/2 F(\xi^0)$$

so that

$$F(\xi^0) \geq \overline{\lim}_{n \rightarrow \infty} F(\xi^n),$$

and by (13),

$$\overline{\lim}_{n \rightarrow \infty} F(\xi^n) = F(\xi^0).$$

Since this is true for every sequence converging to  $\xi^0$ , we have  $\lim F(\xi^n) = F(\xi^0)$ , and the proof is complete.

Theorem 4: Let

$$E = \{ \xi \in \Sigma : y(\xi) \text{ is not unique} \}.$$

Then  $E$  is a countable union of  $N - 2$  dimensional flat spaces in  $\Sigma$ ; in particular  $E$  has  $(N - 1)$  dimensional Lebesgue measure zero.



INTERIM RESEARCH MEMORANDUM  
IRM-31

Proof: We know that  $\xi \in E$  if and only if there is a tie somewhere after we have arranged the weights  $\xi_i \alpha_i (1 - \alpha_i)^{j-1}$ ,  $i = 1, \dots, N$ ;  $j = 1, 2, \dots$  in decreasing order, by Lemma 1. This means that  $\xi \in E$  if and only if we have

$$\xi_i \alpha_i (1 - \alpha_i)^{j-1} = \xi_k \alpha_k (1 - \alpha_k)^{m-1}$$

for some  $i, j, k, m$ ; this completes the proof.

It would appear plausible from this fact that the maximum of  $F(\xi)$ , which yields the value of the game, is achieved at a point  $\xi^*$  for which  $y(\xi^*)$  is almost certainly unique (so that player B has a good pure strategy) but we have no proof of this. There are certainly cases for which player B has no good pure strategy, e.g., if the detection probabilities  $\alpha_i$  are all equal, in which case, we must have  $\xi_1^* = \dots = \xi_N^* = 1/N$ .

To compute (approximately) the solution to a particular game of this type, we can proceed as follows: Find a  $\xi^*$  that yields the maximum of  $F(\xi)$ .

[For each  $\xi$ , we know how to find an optimal  $y(\xi)$ ; then  $F(\xi) = (\xi, s(y(\xi)))$ . The maximum of the function  $F(\xi)$  can be found by some variant of the steepest ascent method.] Express  $\xi^*$  as a convex combination of  $\xi^j$ ,  $j = 1, \dots, N$ , with  $\xi^j$  close to  $\xi^*$ , e.g., let

$$\xi^j = (1 - \epsilon)\xi^* + \epsilon(0, 0, \dots, 1^{\text{th}}, 0, \dots, 0)$$

with  $\epsilon$  small and positive. Let  $y^j$  be a pure strategy for B, optimal against  $\xi^j$ . Then  $\{s(y^j)\}$ ,  $j = 1, \dots, N$ , will approximately "straddle" the critical point  $u^*$  in  $S^*$ . Let

$$M_{ij} = M(i, y^j),$$

$i, j = 1, 2, \dots, N$ . Solve the  $N$ -rowed matrix game with entries  $M_{ij}$ . Let its solution be  $\xi^0, \eta^0, v^0$ . Then  $v^0$  is an upper bound for  $v$ ,  $\eta^0$  is an approximate solution for B, and  $\xi^0$  and  $\xi^*$  are each possible mixed strategies for A. Take  $\xi^0$  or  $\xi^*$ , whichever yields the larger  $F(\xi)$ .

**INTERIM RESEARCH MEMORANDUM**  
**IRM-31**

References: 1. Blackwell, D. and M. A. Girshick. Theory of Games and Statistical Decisions, John Wiley, N. Y., 1954.